Relativity Diagram

I recall reading a book a while back called **Relativity Visualized** by Lewis Carroll Epstein. In it he has his explanation for why nothing can exceed the speed of light. The reason is that everything travels at the speed of light, whether it be purely through space (photons) or purely through time (for a stationary frame of reference, and we multiply time axis by c to get units of space), or some combination of the two. He has an interesting diagram which shows what he means by this concept. Recently on YouTube on the FermiLab channel, the host, physicist Don Lincoln, used the same argument to explain the speed of light, but cautioned that it was just a rough intuitive explanation. The real explanation involves Minkowski diagrams. This paper will explore this concept and diagram, and see how far I can take the diagram as an explanation of the results of special relativity. I will tie it back to the Minkowski diagram, and show how some improvements to the diagram can be made.

The space-time interval squared is defined as $\Delta s^2 = \Delta x^2 - c^2\Delta t^2$. Any two frames of reference will agree on this interval. So we can equate the two frame S and S'.

$\Delta x^2 - c^2\Delta t^2 = (\Delta x')^2 - c^2 (\Delta t')^2$

Then we swap terms to get:

(2) $\Delta R^2 = \Delta x^2 + c^2 (\Delta t')^2 = (\Delta x')^2 + c^2\Delta t^2$

Where R is the hypotenuse of the the triangle formed by Δx and $c\Delta t'$, and also $\Delta x'$ and $c\Delta t$. Because of time dilation

$$\Delta t' = \sqrt{1 - v^2/c^2}\Delta t$$

Substituting that into Equation (2) gives

$$\Delta R^2 = \Delta x^2 + c^2(1 - v^2/c^2)\Delta t^2$$

Since the S' reference frame is moving at speed v with respect to S, we get $\Delta x = v\Delta t$.

So

$$\Delta R^2 = v^2\Delta t^2 + c^2(1 - v^2/c^2)\Delta t^2 = c^2\Delta t^2$$

So we get $\Delta R = c\Delta t$, and

$$\frac{\Delta R}{\Delta t} = c$$

A similar argument is made for $\Delta x'$ and $c\Delta t$. This shows that in any 'frame' of reference where the time axes are swapped, we get objects moving through 'space-time' at the speed c.

This is the basis of the diagram in the book **Relativity Visualized**. I will describe some of the basic relativity results that one can derive from this diagram. Later I will delve more deeply into what this diagram means and how it ties into the formulas of relativity.

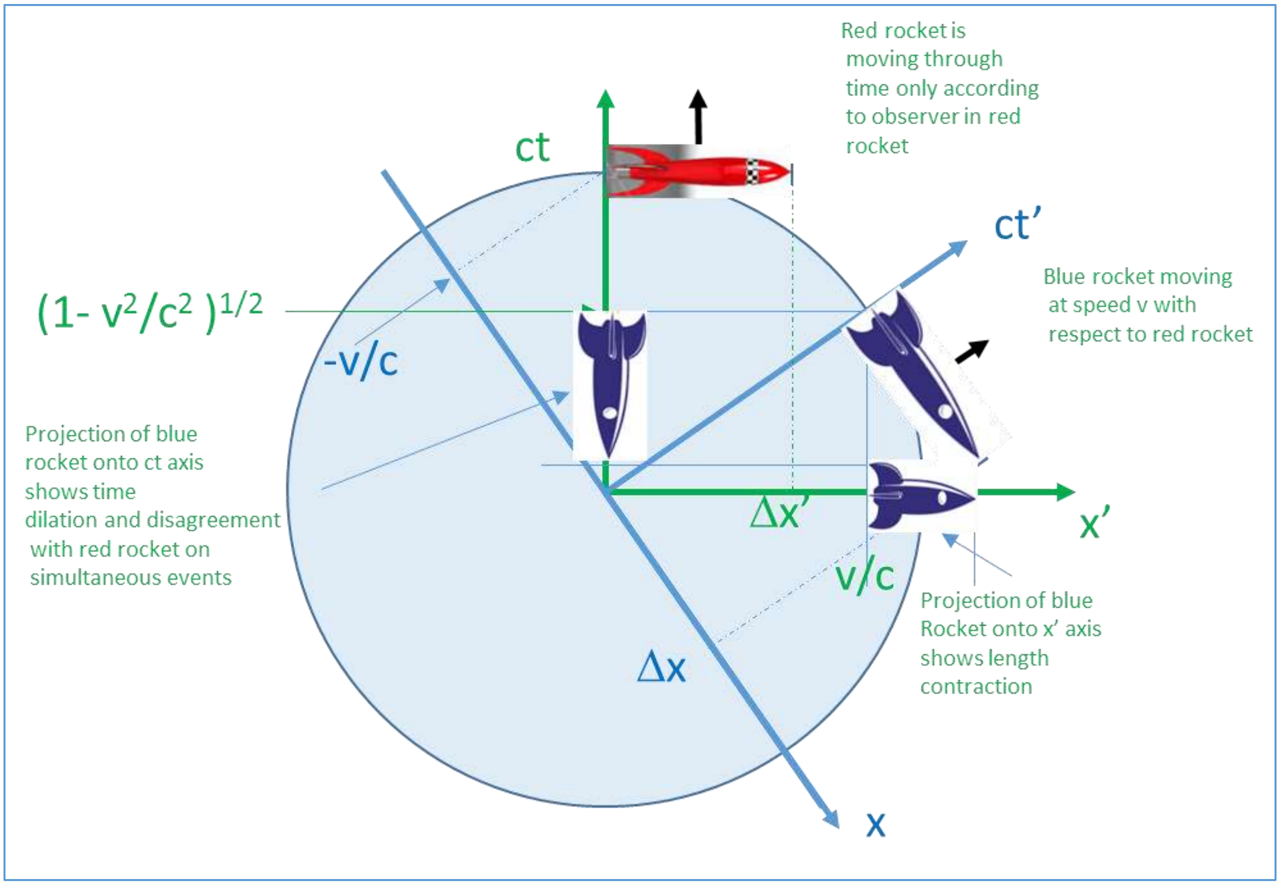

Figure 1: The diagram showing the blue rocket frame moving at speed **v** with respect to the red rocket frame

What this diagram shows is that if we pair **x** with **ct’**, and **x’** with **ct**, we can think of frames of reference that are moving with respect to each other as a standard rotation. The two rockets of equal length (in their respective rest frames) in the diagram start at **t = t’ = 0** at the origin. We name them ‘Red Rocket’ and ‘Blue Rocket’. After one unit of time (according to each frames clock), Red Rocket and Blue Rocket have reached the edge of the circle. Red Rocket measures that Blue Rocket has partially travelled through space and partially through time. Red Rocket measures that it has travelled purely through time. Likewise Blue Rockets measures that it has travelled through time only, and Red Rocket has travelled through space and time.

From Red Rocket’s perspective, the proportion of speed Blue Rocket travels through space is **v/c** (that is, the projection of Blue Rocket’s space interval onto Red Rocket’s space axis). The proportion that Blue Rocket has travelled through time is $\mathbf{(1 - (v/c)^2)^{1/2}}$. Let’s see what kind of relativity results we can derive from the diagram.

If we project Blue Rocket onto the **x’** axis (Red Rocket’s axis), it appears to be contracted by a factor of $\mathbf{(1 - (v/c)^2)^{1/2}}$. This matches what we expect from the Lorentz Transformation formulas of length contraction in the direction of travel. If we project the Red Rocket on the **x** axis (Blue Rocket’s axis), the same contraction is measured in the Blue Rocket frame. Likewise, the projection of the Blue Rocket time axis (**ct’**) against the Red Rocket time axis (**ct**) shows that one unit of time in the Blue Rocket frame is slower in the Red Rocket frame by the same factor $\mathbf{(1 - (v/c)^2)^{1/2}}$. This is also showing that a moving time frame is slower than a stationary one. This diagram indicates that length contraction and time dilation are real in one sense, but apparent in another sense in that they are the result of disagreement over what is simultaneous in each frame of reference. Each frame of reference is measuring the projection of the other rocket onto its axes.

Let's derive these two results directly from the Lorentz Transformations.
S' frame in terms of S frame:
$x' = \gamma (x - (v/c)(ct))$
$ct' = \gamma (ct - (v/c)(x))$

S frame in terms of S' frame:
$x = \gamma (x' + (v/c)(ct'))$
$ct = \gamma (ct' - (v/c)(x'))$

Differential form:
S' frame in terms of S frame:
$\Delta x' = \gamma (\Delta x - (v/c)(c\Delta t))$
$c\Delta t' = \gamma (c\Delta t - (v/c)(\Delta x))$

S frame in terms of S' frame:
$\Delta x = \gamma (\Delta x' + (v/c)(c\Delta t'))$
$c\Delta t = \gamma (c\Delta t' - (v/c)(\Delta x'))$

Length contraction as measured in the S frame:
S measures the length of a moving object by comparing two positions where the two events are simultaneous in S. So **S** chooses formula $\Delta x' = \gamma (\Delta x - (v/c)(c\Delta t))$ and sets $\Delta t = 0$. So $\Delta x' = \gamma (\Delta x)$.
$\Delta x = (1 - (v/c)^2)^{1/2} \Delta x'$.

Time dilation as measured in the S frame:
S measured the time passing in the S' frame by monitoring a moving clock that is at rest in S'. So S chooses $c\Delta t = \gamma (c\Delta t' - (v/c)(\Delta x'))$ because we need to set $\Delta x' = 0$. $\Delta t = \gamma (\Delta t')$. In order for this equation to be true, $\Delta t'$ must be smaller than Δt.

Similar arguments can be made for length contraction and time dilation measured in the S' frame of length and time intervals in S.