Methods of Integration: Mobius Transformations

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There are three general types of definite integrals for which the complex contour integration methods are useful. Integrals of functions of cosines and sines over the interval [0, 2π] are transformed to the complex unit circle. The second and third type of integrals are over the intervals [-∞,∞] and [0, ∞], and are solved by including the interval on the real axis as part of the contour of integration in the complex plane.

This article works out an example of a complex contour integration method that I have not seen in my textbooks or in online articles I have read. I don't know if this is a novel method or not. Maybe someone can point me to the relevant literature. The method transforms an infinite integral on the real line from –∞ to ∞ to a circle in the complex plane using Mobius transformations.

Take for example the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$$

Create a mapping, for example 0 to 1, 1 to ***i***, ∞ to -1 and calculate the associated Mobius transform. This particular mapping is a transform of the x axis to the complex unit circle centered at the origin. See Figure 1.

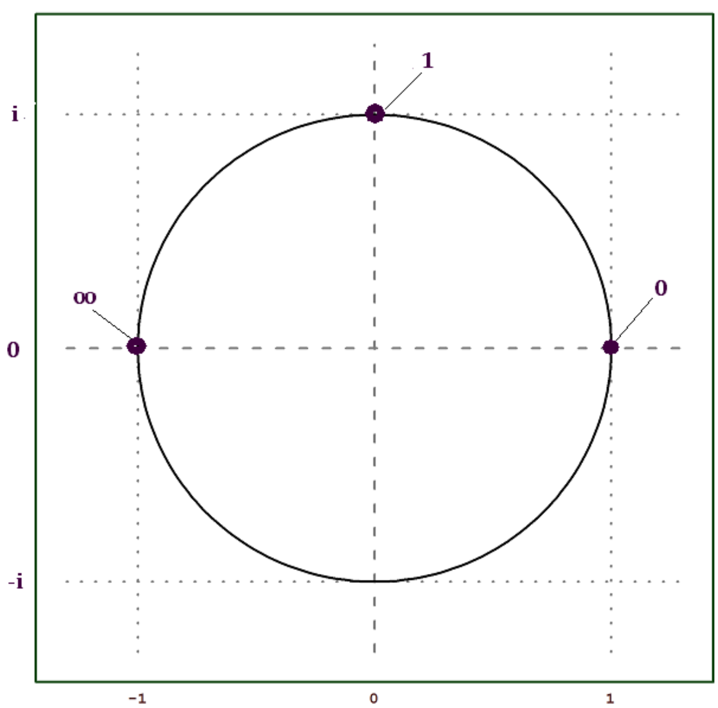

Figure 1. Mapping of 3 points to 3 points.

Three points to three points in the complex plane defines a unique Mobius transformation. Since we are going to map the point at infinity, the best way to calculate with infinity is to use homogeneous coordinates. To read a good explanation of Mobius transformations with homogeneous coordinates and the transform as a 2 x 2 matrix, see the book *Visual Complex Analysis* by *Tristan Needham*, *pp. 154-158*.

The math software I use for numerical results is Euler Math Toolbox (EMT), which is a Matlab type language. To model the point at infinity use homogeneous coordinates, as mentioned above. So

[0, 1, ∞] = [0, 1, 1] / [1,1,0].

However, we cannot directly divide the two vectors because of the zero in the denominator, so we keep numerator and denominator as the first and second rows of a matrix.

$$Z = \begin{bmatrix} Z1 \\ Z2 \end{bmatrix} \text{ and } X = \begin{bmatrix} X1 \\ X2 \end{bmatrix}$$

```
>X1=[0,1,1], X2=[1,1,0], Z1=[1,I,-1], Z2=[1,1,1]
 [0,  1,  1]
 [1,  1,  0]
 [ 1+0i   0+1i  -1+0i  ]
 [1,  1,  1]
```

I created a function in EMT called **`mobiustransform(X1,X2,Z1,Z2)`** that finds the matrix transforms for X to Z and Z to X. It returns two matrices that are inverses of each other. The program is shown at the end of this article. It is based on using the cross-ratio, described in the same book *Visual Complex Analysis*.

```
>{ZtoX, XtoZ} = mobiustransform(X1, X2, Z1, Z2);

>ZtoX
          -1-1i            1+1i
           1-1i            1-1i
>XtoZ
       -0.25+0.25i       0.25+0.25i
        0.25-0.25i       0.25+0.25i
```

Now revert back to non-homogeneous coordinates.

The transform between z and x is:

$$x = \frac{(-1-i)z+1+i}{(1-i)z+1-i}.$$

And

$$dx = \frac{-2i}{(z+1)^2}dz$$

Using these substitutions in the integral, we get the integrand:

$$\frac{-2i/(z+1)^2}{1+(((-1-i)z+1+i)/((1-i)z+1-i))^2}$$

Which simplifies to (I used *Wolfram Alpha* to save time and mistakes):

$$\frac{-i}{2z}$$

The residue is *–i/2*, so the integral equals π, which is the correct answer. In later articles I will solve some more complicated examples, and see if there is a connection between the eigenvalues of the transforming matrix and the location of the poles, or of the residue values.

Here is the function I created in EMT for calculating the Mobius transform with homogeneous coordinates.

```
function mobiustransform(X1,X2,Z1,Z2)
## calculate the Mobius transform and its inverse given a 3 points to 3 points
mapping
      A1 = [X1[2]*X2[1]*X2[3]-X1[3]*X2[1]*X2[2],-X1[1]*(X1[2]*X2[3]-X1[3]*X2[2])];
      A2 = [X1[2]*X2[1]*X2[3]-X1[1]*X2[2]*X2[3],-X1[3]*(X1[2]*X2[1]-X1[1]*X2[2])];
      B1 = [Z1[2]*Z2[1]*Z2[3]-Z1[3]*Z2[1]*Z2[2],-Z1[1]*(Z1[2]*Z2[3]-Z1[3]*Z2[2])];
      B2 = [Z1[2]*Z2[1]*Z2[3]-Z1[1]*Z2[2]*Z2[3],-Z1[3]*(Z1[2]*Z2[1]-Z1[1]*Z2[2])];

      A = A1_A2;
      B = B1_B2;
      ZtoX =A\B;
return { ZtoX, inv(ZtoX) }
endfunction
```