A Relationship between z(3) and z(6) using Fourier series
By Paul Mayer, 6-29-2014

The Riemann zeta function is defined for Re(s) > 1 as

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \tag{1}$$

Using Fourier series to find ζ(2) is a method I have seen in articles I have read online. Find ζ (2n) is straightforward with Fourier series, where n is a positive integer. Start with

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi}{2} - \frac{x}{2} \tag{2}$$

It can be shown with the usual integration method, and here is the graph of the Fourier series to 10 000 terms over the interval [0, 2π] that I generated in Euler Math Toolbox. See this June 14, 2014 posting I wrote for the sample program.

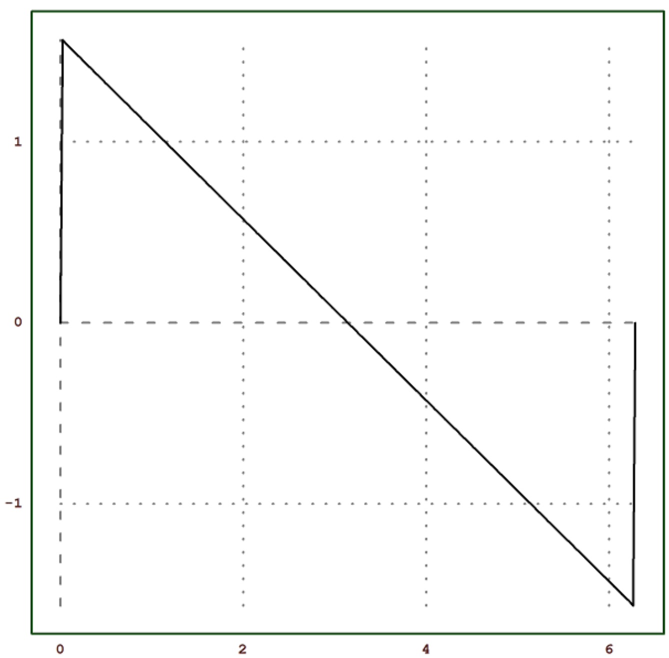

Figure 1. Fourier series with 10 000 terms

To find ζ(2), integrate both sides of (2)

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{x^2}{4} - \frac{\pi x}{2} + C_2 \qquad (3)$$

To find the constant term C_2 note that each term of the Fourier series averages to 0 over the interval [-π, π]. Since (3) is an even function, the same can be said for the interval [0, π].

$$\int_0^{\pi} \left[\frac{x^2}{4} - \frac{\pi x}{2} + C_2\right] dx = 0 \qquad (4)$$

$$\frac{\pi^3}{12} - \frac{\pi^3}{4} + C_2 \pi = 0 \qquad (5)$$

Solving for C_0 we get:

$$\frac{\pi^3}{12} - \frac{\pi^3}{4} + C_2 \pi = 0 \qquad (6)$$

$$C_2 = \frac{\pi^2}{6} \qquad (7)$$

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6} \qquad (8)$$

We now have ζ(2) by substituting x = 0 into (8):

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \qquad (9)$$

To find ζ(4) we follow the same procedure of integrating the series. Integratiing (8) gives

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3} = \frac{x^3}{12} - \frac{\pi x^2}{4} + \frac{\pi^2 x}{6} \qquad (10)$$

Note that the constant of integration is zero because each term of the Fourier sine series is zero at x = 0. Now integrate one more time.

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^4} = -\frac{x^4}{48} + \frac{\pi x^3}{12} - \frac{\pi^2 x^2}{12} + C_4 \qquad (11)$$

The same reasoning applies as for the case of finding C_2.

$$\int_0^{\pi}\left[-\frac{x^4}{48}+\frac{\pi x^3}{12}-\frac{\pi^2 x^2}{12}+C_4\right]dx=0 \tag{12}$$

$$-\frac{\pi^5}{240}+\frac{\pi^5}{48}-\frac{\pi^5}{36}+C_4\pi=0 \tag{13}$$

$$C_4=\frac{\pi^4}{90} \tag{14}$$

Similar to (9) we substitute x = 0 into (11) to get $\zeta(4)$:

$$\sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\pi^4}{90} \tag{15}$$

This procedure can be used to determine the values of zeta for all even integers. The question is, what is the analogous procedure to find zeta for odd integers? I did find (numerically in Euler Math Toolbox using least square fitting) the following result up to a constant:

$$\sum_{k=1}^{\infty}\frac{\cos(kx)}{k}=\log\left(\sin\left(\frac{x}{2}\right)+C \quad for\ x\ in\ interval\ [0,\pi]\right. \tag{16}$$

If the RHS could be integrated in closed form, the problem would be solved. Unfortunately it is not a function that can be integrated in closed form (at least according to Wolfram Alpha). It only has a series representation, which for x evaluated at zero gives the series definition of $\zeta(s)$, where s is an odd integer. So we have not gained anything with this approach.

Is this the end of the story for this method? There is one thing we can still learn about $\zeta(3)$ and the other odd integer arguments. We can find a connection between $\zeta(n)$ and $\zeta(2n)$ using the Fourier Series approach.

To see how, take Equation (8) and square both sides.

$$\left[\sum_{k=1}^{\infty}\frac{[\cos(kx)]}{k^2}\right]^2=\left[\frac{x^2}{4}-\frac{\pi x}{2}+\frac{\pi^2}{6}\right]^2 \tag{17}$$

The LHS is given by:

$$\left[\sum_{k=1}^{\infty}\frac{[\cos{(kx)}]}{k^2}\right]^2=\sum_{k=1}^{\infty}\frac{\cos{(kx)}^2}{k^4}+\sum_{k=1}^{\infty}\sum_{m\neq k}^{\infty}\frac{\cos(kx)\cos{(mx)}}{k^2m^2} \tag{18}$$

The RHS is given by:

$$\left[\frac{x^2}{4}-\frac{\pi x}{2}+\frac{\pi^2}{6}\right]^2=\frac{x^4}{16}-\frac{\pi^1x^3}{4}+\frac{\pi^2x^2}{3}-\frac{\pi^3x^1}{6}+\frac{\pi^4}{36} \tag{19}$$

Integrating (18) from [0, π] the cross terms of the series evaluate to zero, so we are left with

$$\int_0^{\pi}\left[\sum_{k=1}^{\infty}\frac{\cos{(kx)}^2}{k^4}\right]dx=\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^4} \tag{20}$$

Integrating (19) from [0, π] we get

$$\int_0^{\pi}\left[\frac{x^4}{16}-\frac{\pi^1x^3}{4}+\frac{\pi^2x^2}{3}-\frac{\pi^3x^1}{6}+\frac{\pi^4}{36}\right]dx=\frac{\pi^5}{180} \tag{21}$$

Equating (20) and (21) we get ζ(4)

$$\sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\pi^4}{90} \tag{22}$$

So we see that we can derive the ζ(2n) given the Fourier series and its polynomial equivalent for any integer ζ(n) if there is a polynomial equivalent to the Fourier series.

This result also applies to odd n. So if there is a Taylor series representation of the Fourier series for n = 3, we know how it relates to ζ(6).

$$\sum_{k=1}^{\infty}\frac{\cos(kx)}{k^3}=\zeta(3)+c_1x^1+c_2x^2+c_3x^3+\cdots \tag{23}$$

We know that the constant in the Taylor series is ζ(3) by substituting x = 0 in (23). Note that for (8) and (11) the sum of the powers in each term for 'π' and 'x' are 2 and 4 respectively. In the general case for even ***n*** the sum of the powers is ***n***. Assume that the same rule applies to odd ***n***, except now the Taylor series has an infinite number of non-zero terms. So we rewrite (23) as

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^3} = \zeta(3) + \sum_{k=1}^{\infty} a_k \pi^{3-k} x^k \tag{24}$$

Using the squaring method shown on page 3 and page 4, we obtain from the LHS of (24)

$$\left[\sum_{k=1}^{\infty} \frac{[\cos(kx)]}{k^3}\right]^2 = \sum_{k=1}^{\infty} \frac{\cos(kx)^2}{k^6} + \sum_{k=1}^{\infty} \sum_{m \neq k}^{\infty} \frac{\cos(kx)\cos(mx)}{k^3 m^3} \tag{25}$$

The RHS is

$$\left[\zeta(3) + \sum_{k=1}^{\infty} a_k\, \pi^{3-k} x^k\right]^2 = \zeta(3)^2 + \sum_{k=1}^{\infty} \zeta(3) a_k\, \pi^{3-k} x^k + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \left(a_k\, \pi^{3-k} x^k\right)\left(a_m\, \pi^{3-m} x^m\right) \tag{26}$$

From (25) we obtain

$$\int_0^{\pi} \sum_{k=1}^{\infty} \frac{\cos(kx)^2}{k^6} dx = \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{k^6} \tag{27}$$

From (26) we obtain

$$\int_0^{\pi} \left[\zeta(3)^2 + \sum_{k=1}^{\infty} \zeta(3) a_k\, \pi^{3-k} x^k + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \left(a_k\, \pi^{3-k} x^k\right)\left(a_m\, \pi^{3-m} x^m\right)\right] dx = \frac{\pi^7}{1890} \tag{28}$$

We know that (28) is true because we know that (27) equals $(\pi/2)\, \zeta(6)$.

From (28) it seems reasonable that

$$\zeta(3) = a_0 \pi^3 \tag{29}$$

where the constant a_0 is probably irrational (because no numerical searches have yet found a rational coefficient, and theoretical results indicate it to be highly unlikely). It seems reasonable that the constant a_0 has the square root of some rational number in it, since all the terms in (28) add up to a rational number times π^7. This is not necessary however, just a guess.