Desert Island Formulas

Paul Mayer, June 20, 2014

In anticipation of ever being stuck on a deserted island, and wanting to have a minimum of math formulas to remember (from which other formulas can be derived), what would they be? Let's say that you expect to spend a few years on the island before getting rescued, and your job when you get home depends on math, and you don't want your math knowledge to get rusty. Here are a couple candidates for useful formulas.

Trigonometric Formulas

The following equation can be used to find many trig identities, and also to derive what many mathematicians consider to be the most beautiful formula in math.

$$e^{ix} = \cos(x) + i\sin(x) \qquad (1)$$

Let's say you want to find an identity for cos(ax + by).

$$e^{i(ax+by)} = e^{iax}e^{iby} = [\cos(ax) + i\sin(ax)]\,[\cos(by) + isin(by)] \qquad (2)$$

$$e^{i(ax+by)} = [\cos(ax+by) + isin(ax+by)] \qquad (3)$$

$$[\cos(ax) + i\sin(ax)][\cos(by) + isin(by)] =$$

$$= \cos(ax)\cos(by) - \sin(ax)\sin(by) + i\,[\cos(ax)\sin(by) + \sin(ax)\cos(by)\,] \qquad (4)$$

From (3) and (4), matching the real part to real part, and imaginary part to imaginary part, we get

$$\cos(ax + by) = \cos(ax)\cos(by) - \sin(ax)\sin(by) \qquad (5)$$

$$\sin(ax + by) = \cos(ax)\sin(by) + \sin(ax)\cos(by) \qquad (6)$$

So we get two formulas for the price of one.

A formula that I never remember is for rotating points in the xy plane by a given angle θ. I can however derive it easily by converting point (x_0, y_0) into a complex number $x_0 + iy_0$ and multiplying by $e^{i\theta}$.

$$(x_0 + iy_0)\,e^{i\theta} = (x_0 + iy_0)(\cos(\theta) + i\sin(\theta)) \qquad (7)$$

$$x + i\,y = x_0\cos(\theta) - y_0\sin(\theta) + i\,[x_0\sin(\theta) + y_0\cos(\theta)] \qquad (8)$$

The real part of (8) is the new x value, and the imaginary part of (8) is the new y value for the rotated point.

Now for the elegant formula that combines the 'big five' constants in to one simple expression, and also the three operations-- addition, multiplication, exponentiation.

Let $x = \pi$ in the original equation (1).

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 \qquad (9)$$

From (9) we immediately get

$$e^{i\pi} + 1 = 0$$

Contemplation of this formula will help wile away some of the lonely days on the desert island.

Volume and Area Formulas

Knowledge of one volume formula for right circular cones will allow one to derive formulas for spheres and circles in any dimension. The volume of a cone in N dimensions is related to the cross-sectional sphere of N-1 dimensions. For example, for a 3-dimensional cone, the area and circumference of the circular cross-section at radius R can easily be found with the aid of a little calculus. Likewise for higher dimensional cases. For a 3-dimensional sphere, it will be the volume and surface area that are calculated.

Starting with right circular cone in N dimensional space, the volume of the cone is

$$V = \frac{1}{3}\pi R^N \qquad (10)$$

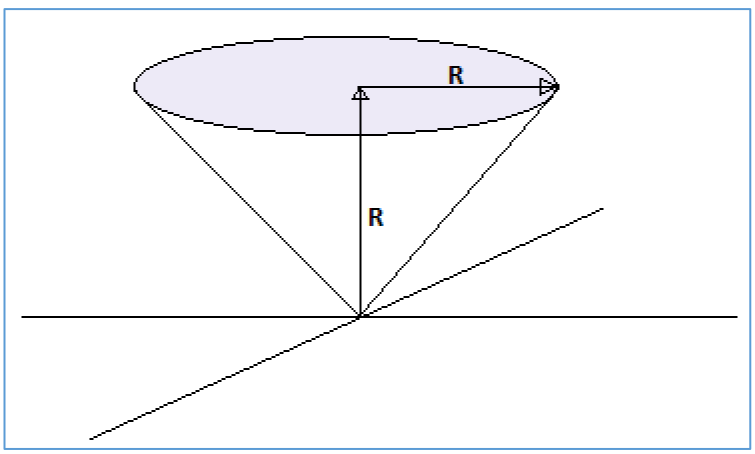

Figure 1. 3 dimensional right circular cone

For example, the volume of the 3 dimensional cone with height R and radius R is

- 3D cone volume = $1/3\pi R^3$.
- The first derivative is πR^2, the area of the sectional circle at radius R.
- The second derivative is $2\pi R$, the circumference of the sectional circle at radius R.
- The third derivative is 2π. What is the geometric interpretation, other than the diameter of the unit circle?

For a 4 dimensional right circular cone, the volume is

- 4D cone volume = $1/3\pi R^4$.
- The first derivative is $4/3\pi R^3$, the 3D volume of the sphere which is the section of the cone at radius R.

- The second derivative is $4\pi R^2$, the 2D surface area of the sectional sphere at radius R.
- The third derivative is $8\pi R$. What is the geometric interpretation, other than it is 4 times the circumference of the section through the center of the sphere?
- The fourth derivate is 8π, which has what interpretation?

So one can see the pattern. The first two derivatives have an obvious geometric interpretation. These results can be extended to any dimension.

One can see the possibility of generalizing these results to cones (pyramids) of arbitrary cross sectional shape, but that will be another article.

So from two key equations, it is possible to derive others. The only two formulas to remember are (1) and (10).